Weak measurements and their implications in quantum foundations

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Abstract: In this dissertation first the basics of Weak measurements are reviewed. Then we shall see how these relatively new tools give insight into the deep problems of physics, by critically studying the cases in the literature where the technique was applied to some foundational issues in quantum mechanics. In particular we focus on the possibility of having counter-factual arguments in quantum theory and the reality of wavefunction. The arguments in this dissertation are supported by the most recent experiments done in laboratories. Finally at each section the current relevant criticisms are discussed.

"Perhaps I did something to rekindle interest in these questions. People who are younger than me now tend to agree that there are problems to be solved. Of course, most of them dont tackle these problems. They rather work on lines in elementary-particle physics like string theory. But they are generally more open to the idea that there are problems with the foundations of the quantum theory than their teachers were."

Jeremy Bernstein , *"John Stewart Bell: Quantum Engineer,"* in Quantum Profiles (Princeton University Press, Princeton, 1991). p86

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1 Introduction

Measurement plays an important role in quantum mechanics. A lot of foundational issues in quantum mechanics arise from our interpretation of measurement results and the system subject to measurement. For an introductory review of these issues one might want to consult [2]. Most of the technical issues in applications of quantum mechanics can be resolved by treating the measuring device as an external system, which behaves classically and causes the collapse of wavefunction. However, the representational completeness postulate of quantum mechanics requires that any isolated physical system can be described with a wavefunction in a complex Hilbert space. Therefore, it is the choice of the physicist to consider the measuring device as an external potential or a dynamical coupling which is described by the theory as an internal system. To enable us to apply the theory of quantum mechanics to the measurement procedure (and hence being able to define the role of measurer) von Neumann formulated an interaction which describes measurement through the Hamiltonian [16]

$$H = H_0 + g(t)(q \otimes I)(I \otimes A) + H_a \tag{1}$$

where H_0 is the Hamiltonian which causes the evolution of the system in the absence of measurement, H_a is the Hamiltonian under which influence the isolated apparatus evolves, q is the observable of apparatus (*e.g.* coordinates of the pointer). Here A is the observable of the system which is subject to measurement. $g(t)$ represents the the coupling function which is only non-zero in the time interval $[0, T]$, with T being the duration of the measurement. Letting $g(t) = g_0 f(t)$, where g_0 is a coupling constant and $\int_0^T f(t)dt = 1$, then the state of the measured and measurer system combined, $|\Phi\rangle$, obey Schroedinger equation

$$i\hbar\frac{\partial|\Phi\rangle}{\partial t} = H|\Phi\rangle. \tag{2}$$

The initial state of the combined system can be expanded in the normalised eigenstates of the observable, A, and the normalised state of the measuring device as a product state

$$|\Phi(0)\rangle = |\alpha\rangle|\Psi\rangle = |\alpha\rangle\sum_i c_i|\Psi_i\rangle. \tag{3}$$

The combined system evolves with a time evolution $U = \exp(-\frac{i}{\hbar}\int_0^T Hdt)$ which under the impulsive conditions can be approximated as

$$|\Phi(T)\rangle \approx \sum_i c_i \exp\Big(-\frac{i}{\hbar}g_0 q a_i\Big)|\alpha\rangle|\Psi_i\rangle \tag{4}$$

This shifts the operator conjugate to q (momentum, π, in this case) by the value $g(t)A$ as can be seen in the Heisenberg picture

$$\frac{d}{dt}\pi = -ig(t)A \tag{5}$$

Note that q could be any observable of the measuring device. The reason we have chosen to discuss the case of position is the ease of use in describing the Stern-Gerlach experiment in the later sections. One could alternatively choose apparatus' pointer momentum and observe the shift in position.

As it can be seen in equation (4) the momentum gains different values for each eigenstate of the observable A, causing an entanglement between the measured system and the measurer

$$|\Phi(T)\rangle = \sum_i c_i|\alpha_i\rangle|\Psi_i\rangle \tag{6}$$

where $|\alpha_i\rangle$ is the shifted pointer state corresponding to the ith eigenstate of the observable A. According to the picture given above, the combined system undergoes a deterministic and continuous evolution, causing the measuring device to be in a superposition state of non-overlapping functions; the state which is not normally seen in the lab and is indeed inconsistent with the picture given by the external treatment of the measurer. This inconsistency is called the "Measurement Problem". However, this is a paradox, if the wavefunction is interpreted as a mathematical function with one-to-one correspondence with the physical systems. There are many responses to the problem. The realist approach is discussed later in this article. Nevertheless, the dominant view ignores the problem by considering the wavefunction as the mathematical description of our maximal *knowledge* of the system. Then it is assumed that it's not surprising that *observing* the system will change the wavefunction, causing the collapse. This view tells us that there is discontinuous and objectively probabilistic collapse to only one term in the summation with probability $|c^2|$. With the picture of quantum measurement as a physical process, one can explore a range of interesting phenomena, which are testable and surprising; they challenge our "quantum common sense". In the next section we shall see the evolution of an ensemble of selected systems under weakened measurement interaction. Later we shall explore the consequences of measuring a single-particle system in the weak conditions, but done in the adiabatic limit with protection.

2 Weak measurement

Weak measurement is a modified form of the von Neumann measurement process introduced in (1), on a "post-selected" ensemble of identically prepared particles with a weak interaction or increase the uncertainty in the observable, q. The weakened measurement interaction causes a negligible disturbance to the quantum system. However, consequently almost no information can be retrieved from a single impulsive measurement, because the shift in the position of the pointer would be less than the uncertainty in that position. Therefore, one needs to repeat the experiment many times to gain valuable information.

Let the position of the pointer of the measuring device, q, be represented by a Gaussian function with a spread

Δ,

$$\langle q|\alpha(t=0)\rangle \propto \exp[-(q/2\Delta)^2] \tag{7}$$

where $|\alpha\rangle$ represents the state of the measuring device. For simplicity we assume that the combined system evolves solely due to the second term of the equation (1). Post-selection the state of the quantum system in $|\Psi_2\rangle$ gives the state of the measuring device at the end of the process as

$$\begin{aligned}|\alpha(t=T)\rangle &= \langle\Psi_2|\exp(-iqA)|\Psi_1\rangle\exp[-(q/2\Delta)^2] \\ &= \langle\Psi_2|\Psi_1\rangle\Big(\exp(-iqA_w) \\ &+\sum_{n=2}^{\infty}\frac{(iq)^n}{n!}[(A^n)_w-(A_w)^n]\Big)\exp[-(q/2\Delta)^2]\end{aligned} \tag{8}$$

where

$$A_w=\frac{\langle\Psi_2|A|\Psi_1\rangle}{\langle\Psi_2|\Psi_1\rangle}. \tag{9}$$

A_w is called weak value, which is the result we obtain from weak measurements. These weak values appear on the apparatus, by the shift cause in the conjugate variable to q. By making the Δ small enough the uncertainty in the momentum of the pointer gets larger and interaction gets weaker. Hence the quantum system does not get disturbed. We shall clarify this picture by bringing examples of this type of measurement in the following sections. The details of technical tools and a response to some concerns about the meaning of weak values which was raised at the early years of formulation of the theory could be found in [18].

Weak measurements, following their successful experimental results [13, 12, ?, ?, 14] have received significant attention. They can be used in measurement of joint and non-local observables. With the aid of weak measurements one can identify multipartite entanglement which is a building block of cluster state quantum computation. However, besides their advantage for applied physics, one can use them at a very foundational level. In the next section we shall see how the weak measurements could be used to revisit solid arguments about counter-factuality of laws of physics.

2.1 Weak test of Hardy's paradox and possibility of realist approaches to quantum theory

Hardy's paradox is a thought-experiment which was originally designed to demonstrate that local realist quantum theories are paradoxical [9]. It is widely believed that the resolution to the paradox is giving up the counter-factual arguments. Counter-factual reasoning is one which refers to deduction based on elements which are not direct result of basic measurements. Nevertheless, as we shall see here, using weak measurements one can resolve the paradox and furthermore find a deeper logical structures of quantum theory.

Hardy's thought experiment consists of two overlapping Mach-Zehnder interferometers as shown in figure (1). One is for electron and the other one for positron. One arm of the electron interferometer overlaps with the positron interferometer at a point, P. For separated interferometers the particles will only be detected at $C^{\pm}$. However, in the overlapping interferometers, if an electron and a positron both enter the overlapping arm, denoted with $u^{\pm}$, they will annihilate with probability equal to 1. This is expressed as $|u^+\rangle|u_-\rangle \to |\gamma\rangle$, where $|\gamma\rangle$ is the state of radiation due to annihilation. It can be shown that as a consequence of the interaction between the electron and positron the detectors $D^{\pm}$ can click.

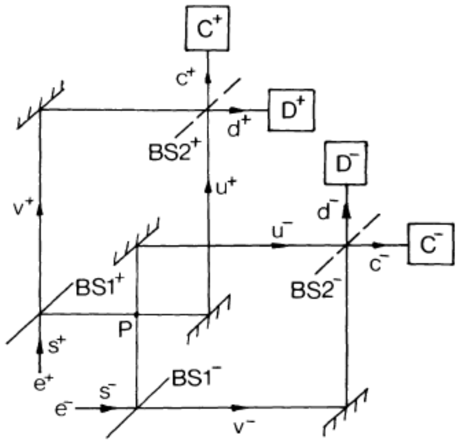

Figure 1: Two overlapping Mach-Zehnder interferometers for electron and positron. The electron goes through the path u^- and the positron goes through u^+ and they overlap at P. [9]

Now let us review the paradox by analysing the experiment. The fist beam splitter, $BS1^{\pm}$, acts on the initial path states as

$$|s^{\pm}\rangle \to (1/\sqrt{2})(i|u^{\pm}\rangle+|v^{\pm}\rangle) \tag{10}$$

Then the operation of $BS2^{\pm}$ on each path is

$$|u^{\pm}\rangle \to (1/\sqrt{2})(|c^{\pm}\rangle+i|d^{\pm}\rangle) \tag{11}$$

and

$$|v^{\pm}\rangle \to (1/\sqrt{2})(i|c^{\pm}\rangle+|d^{\pm}\rangle) \tag{12}$$

If the $BS2^{\pm}$ is removed then paths $u^{\pm}$ go to the $c^{\pm}$ and the paths $v^{\pm}$ will end up in $d^{\pm}$,

$$|u^{\pm}\rangle \to |c^{\pm}\rangle \tag{13}$$

and

$$|v^{\pm}\rangle \to |d^{\pm}\rangle \tag{14}$$

.

Let the particles be prepared in the state $|\Psi_{init}\rangle = |s^+\rangle|s^-\rangle$. Passing through the first beam splitter, this state will evolve into

$$|\Psi_{BS1}\rangle = \frac{1}{2}(i|u^+\rangle+|v^+\rangle)(i|u^-\rangle+|v^-\rangle) \tag{15}$$

After the overlap point, P, if both particles are in the inner arm they will annihilate. Hence the total state would be

$$|\Psi_P\rangle = \frac{1}{2}(-|\gamma\rangle+i|u^+\rangle|v^-\rangle+i|v^+\rangle|u^-\rangle+|v^+\rangle|v^-\rangle) \tag{16}$$

If both the second beam splitters, $BS2^{\pm}$, are removed, then equation (16) will evolve into

$$|\Psi_{\not{B}S2^{\pm}}\rangle = \frac{1}{2}(-|\gamma\rangle + i|c^{+}\rangle|d^{-}\rangle + i|d^{+}\rangle|c^{-}\rangle + |d^{+}\rangle|d^{-}\rangle) \tag{17}$$

Keeping $BS2^{+}$ and removing $BS2^{-}$, however, will make the final state to be

$$|\Psi_{\not{B}S2^{-}}\rangle = \frac{1}{2\sqrt{2}}(-\sqrt{2}|\gamma\rangle - |c^{+}\rangle|c^{-}\rangle + 2i|c^{+}\rangle|d^{-}\rangle + i|d^{+}\rangle|c^{-}\rangle) \tag{18}$$

Or if instead $BS2^{-}$ is kept and $BS2^{+}$ is removed, then the state will be

$$|\Psi_{\not{B}S2^{+}}\rangle = \frac{1}{2\sqrt{2}}(-\sqrt{2}|\gamma\rangle - |c^{+}\rangle|c^{-}\rangle + i|c^{+}\rangle|d^{-}\rangle + 2i|d^{+}\rangle|c^{-}\rangle) \tag{19}$$

And finally with both beam splitters $BS2^{\pm}$ in place we have

$$|\Psi_{BS2}\rangle = \frac{1}{4}(-2|\gamma\rangle - 3|c^{+}\rangle|c^{-}\rangle + i|c^{+}\rangle|d^{-}\rangle + i|d^{+}\rangle|c^{-}\rangle - |d^{+}\rangle|d^{-}\rangle) \tag{20}$$

Now let the element of counter-factuality (particles taking paths which their existence is not inferred from a measurement result) be introduced by the hidden variable λ. Here we adopt the notation that if any of the beam splitters is removed, it is denoted by ∞ and if it is in place, denoted by 0. Using this notation we shall express the instance where the detector D clicks, with the Beam splitter $BS2^{\pm}$ in place as $D(0,\lambda) = 1$ and if it does not click with the same condition with $D(0,\lambda) = 0$ and similar for other combinations. We shall see, in the following, that the assumptions above leads to a so-called paradox. Of course, paradoxes in a self-consistent physical theory are consequences of some wrong assumptions or ignoring some facts which are unknown at the time of formulating the paradox and one cannot accept an inconsistency in the laws of nature. After reformulating the paradox, we shall see a solution due to more recent theoretical advances, later in this section.

From equation (17) we have

$$C^{+}(\infty,\lambda)C^{-}(\infty,\lambda) = 0 \tag{21}$$

Similarly (18) gives

$$\text{if } D^{+}(0,\lambda) = 1 \text{ then } C^{-}(\infty,\lambda) = 1 \tag{22}$$

Also from (19) we know

$$\text{if } D^{-}(0,\lambda) = 1 \text{ then } C^{+}(\infty,\lambda) = 1 \tag{23}$$

And finally (20) requires that

$$D^{+}(0,\lambda)D^{-}(0,\lambda) = 1 \text{ for } \tfrac{1}{16}\text{th of experiments.} \tag{24}$$

One can see the contradiction here. From equations (22) and (23) it is required that for the experiments where $D^{+}(0,\lambda)D^{-}(0,\lambda) = 1$, we must have $C^{+}(\infty,\lambda)C^{-}(\infty,\lambda) = 1$, which is in direct contradiction with (21).

Before popularity of weak measurements, the mainstream tended to resolve the issue by denying prediction about the past of the basic measurement (retrodiction). Nevertheless, weak measurements teach us that retrodictive arguments are indeed valid [4]. It is worth mentioning that weak measurement is not the only solution which contains counter-factual reasoning. Bohm's model discussed in section (3.2) provides another solution by treating the wavefunction and the point-like particle as separate constituents of the quantum object.

Note that the detectors $C^{\pm}$ and $D^{\pm}$, measure the the projectors on the states $\frac{1}{\sqrt{2}}(|u^{\pm}\rangle + |v^{\pm}\rangle)$ and $\frac{1}{\sqrt{2}}(|u^{\pm}\rangle - |v^{\pm}\rangle)$. The case we are studying concerns with the stage after the analysis of point P and in particular, when the particles did not annihilate. Hence we pre-select the state

$$|\Psi_1\rangle = \frac{1}{\sqrt{3}}|v^{+}\rangle|u^{-}\rangle + \frac{1}{\sqrt{3}}|u^{+}\rangle|v^{-}\rangle + \frac{1}{\sqrt{3}}|v^{+}\rangle|v^{-}\rangle \tag{25}$$

where we project out the term $|u^{+}\rangle|u^{-}\rangle$ in equation (15), which corresponds to annihilation. Also we mentioned in the formulation of Hardy's paradox that the inconsistency arises in the experiments where we have $D^{+}(0,\lambda)D^{-}(0,\lambda) = 1$. Therefore we post-select the states

$$|\Psi_2\rangle = \frac{1}{2}(|v^{+}\rangle - |u^{+}\rangle)(|v^{-}\rangle - |u^{-}\rangle) \tag{26}$$

We are interested in studying the paths of the particles to see whether it is necessary to throw out the notion of existence of the particles "when no-one looks at them", or they could have gone through paths and we can confirm it by weak measurement of them. So we need to measure the particles "occupation operators"

$$\begin{aligned} \hat{N}_v^{+} &= |v^{+}\rangle\langle v^{+}|, & \hat{N}_u^{+} &= |u^{+}\rangle\langle u^{+}| \\ \hat{N}_v^{-} &= |v^{-}\rangle\langle v^{-}|, & \hat{N}_u^{-} &= |u^{-}\rangle\langle u^{-}| \end{aligned} \tag{27}$$

The other information we need is the joint measurement of the particles' paths. Again due to the weakness the measurements we can indeed perform the measurements of the joint occupation operators

$$\begin{aligned} \hat{N}_{v,u}^{+,-} &= \hat{N}_v^{+}\hat{N}_u^{-}, & \hat{N}_{u,v}^{+,-} &= \hat{N}_u^{+}\hat{N}_v^{+} \\ \hat{N}_{u,u}^{+,-} &= \hat{N}_u^{+}\hat{N}_u^{-}, & \hat{N}_{v,v}^{+,-} &= \hat{N}_v^{+}\hat{N}_v^{-} \end{aligned} \tag{28}$$

And the act of measurement does not change the wavefunction. Using equation (9) one can calculate the weak values of our pre- post- selected system as

$$N_{u_w}^{-} = 1 \qquad N_{u_w}^{+} = 1 \tag{29}$$

$$N_{v_w}^{-} = 0 \qquad N_{v_w}^{+} = 0 \tag{30}$$

$$N_{u,u_w}^{+,-} = 0 \tag{31}$$

$$N_{u,v_w}^{+,-} = 1 \qquad N_{v,u_w}^{+,-} = 1 \tag{32}$$

$$N_{v,v_w}^{+,-} = -1 \tag{33}$$

We see that these results have solved the paradox. Let us first point out the fact that the results obtained above are both logically consistent and also satisfy the requirements for the experiments. Later we shall see how these echo the counter-factual arguments.

The statement that “the positron(electron) must have gone through the overlapping arm for the $D^{\pm}$ to click and the electron (positron) must have been in the non-overlapping arm” is satisfied by the equations in (32). But, we know that there was only one electron-positron pair in the experiment. The value of (33), however, preserves the consistency. One should not expect to have an equivalence between the classical and quantum intuitions. The physical results are true, as long as they are mathematically and experimentally consistent. One can interpret the minus result as existence of *minus one* pair in the non-overlapping arm. There are different interpretations of negative values. A lot of physicists have interpreted them as negative probability of having real particles. However, the author believes that in the presence of other interpretations, one should not change the meaning of mathematical concepts. It is meaningless to consider negative probabilities in a theory. An operational interpretation of these results seems to be the safest option. As we shall see in the next sections, these values correspond to the evolution of the pointer in the opposite direction, as it is expected from what should be induced by the devices in the lab. Nonetheless, this minus value is crucial in the consistency of our interpretation. From above, we know that the electron did not go through the non-overlapping arm, which is satisfied by $N^{-}_{v_w} = 0$ in (30). However, $N^{+,-}_{u,v_w} = 1$ requires that it was in the non-overlapping arm. The minus sign resolves the issue in

$$N^{-}_{v_w} = N^{+,-}_{u,v_w} + N^{+,-}_{v,v_w} = 1 - 1 = 0. \tag{34}$$

This type of analysis indeed confirms counter-factual arguments in the following sense: Given a pre- and post-selected system, if a strong von Neumann measurement of the observable $\hat{A}$ results in $A = a$ with certainty, the weak and strong values will be the same, *i.e.* $A = A_w$. Consequently, statements about the results of experiments which can be determined under *separate* von Neumann measurement, will remain true for simultaneous weak measurement. Hence there are experiments which talk about the reality of quantum properties, even “when no-one disturbs them”. We shall study the reality of wave-function in depth in section (3.1).

2.2 Experimental test of the argument

Recently weak measurements have got significant attention, due to the new successful experimental results. An experiment, relevant to the argument brought in this article is the ”Joint weak measurement of a photon pair as a probe of Hardy’s paradox” [12]. I shall not go through any details of the experiment. However, the results are given as a support of the argument in the previous sections.

In this experiment horizontal and vertical photons are used instead of positron and electrons in the original thought experiment. We refer to them as P and E photons. Half-mirrors are used as beam splitters. A quantum interference effect is used instead of annihilation. As a consequence of this interference, the E and P photons will be removed if the both enter the inner arm. Otherwise, they will pass through the interferometer without any disturbance. The arms of the interferometer are adjusted such that $85 \pm 3\%$ of the photon pairs in the inner arm are eliminated in the real experiment. Figure (2) illustrates the schematic of the photonic Hardy’s experiment setup.

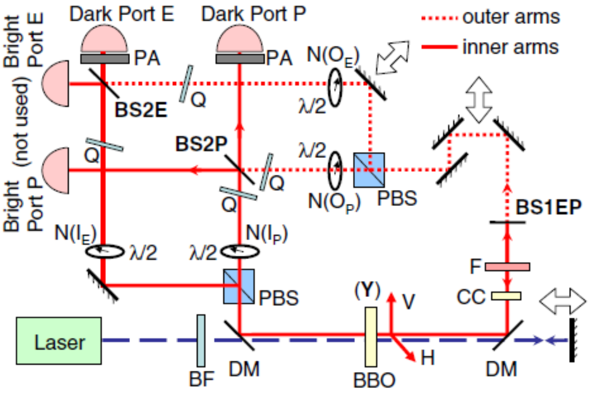

Figure 2: **The experimental setup for the weak measurement of arm occupations N in Hardys paradox.** “An orthogonally polarized collinear photon pair is produced in a BBO crystal. Each photon can be either reflected or transmitted at beam splitter $BS1EP$ (entering the inner or outer arms, respectively). Pair annihilation in the inner arms is achieved by an absorptive switching effect at Y. PBSs then separate the collinear photons into their own interferometers, E and P. Weak measurements of N are induced with half-wave plates ($\lambda/2$) in the chosen arms and read out by PAs at the interferometer dark ports.” [12]

An important part of the experiment which allows us to weakly measure the path of the photon is a half-wave plate at each arm. The procedure is as follows. If the wave plate is placed and aligned in, say, the outer arm of the E photon, the polarisation will rotate by 90°. Hence the polarisation of the photon at the dark port of E interferometer will indicate which path the photon has taken. However, this would be a strong measurement of the occupation operator and strongly disturbs the quantum system. In this case, the interference will be destroyed and the paths will be completely distinguishable. In these types of experiments, weakening is achieved by the polarisation rotation. In this particular case the half-wave plate was rotated by the angle $\theta = 20°$. Now to test the paradox, *i.e.* to see if the photon pairs were in the overlapping arm simultaneously, one needs to measure the joint occupation numbers defined in equation (28). This is done experimentally by placing half-wave plates at each path, just before the final beam splitter. The measurement of the occupation operator of each pair is measured by only rotating only those two wave plates. The experimental results of the weak measurements are tabulated in table (1).

The discrepancies are due to the imperfect photon elimination in the overlapping point and other sources of noise. Nevertheless, the results are significantly consistent with the theory provided in the previous section.

Table 1: The weak values for the occupation operators. [12]

	$N(u_E)$	$N(v_E)$	
$N(u_P)$	$0.245 \pm 0.068[0]$	$0.641 \pm 0.083[1]$	$\mathbf{0.926 \pm 0.015[1]}$
$N(v_E)$	$0.719 \pm 0.074[1]$	$-0.759 \pm 0.083[-1]$	$\mathbf{-0.078 \pm 0.020[0]}$
	$\mathbf{0.924 \pm 0.024[1]}$	$\mathbf{0.087 \pm 0.023[0]}$	

The most interesting part of this confirmation is the negative weak value. In every case, when we have a negative weak value of a measurement, this means that the pointer has been shifted in the opposite direction from what we would expect. For instance in this experiment the photons are found to be rotated with the same angle as induced by the wave plate, but in the opposite direction. This is, as demonstrated above, a consistent picture of the logical structure of quantum theory. Nonetheless, very counter intuitive.

2.3 Weak measurement and the reality of Wavefunction reality

It was pointed out in the introduction section that there has been a tendency in the physics community to consider the wavefunction as a mathematical description of the maximal *knowledge* one can have about the system, not as a real physical object. This view will be challenged if one could find an experimental method to directly measure the wavefunction. Recently, there has been an experiment, using weak measurements that claims to make that challenge[13]. This result is significant and in case of conclusive proof, gives a new meaning to the wavefunction. If the wavefunction is directly measurable, then one can no longer consider it as an abstract object, representing the probability of *finding* a particle in a certain state. But one has to interpret it as a physically real object.

As we shall see in the following as a result of this experiment, one can see the real and imaginary part of the wavefunction, appearing on the measuring apparatus. This result should not be surprising if one takes the meaning of weak values seriously.

The experiment was performed on transverse spatial wavefunction of a single photon and claimed to be generalisable to degree of freedom of quantum particles. It uses the fact that weak measurements do not disturb the wavefunctions and hence strong measurements of observables can be done following a sequence of weak measurements of other observables complimentary to the observables of the first measurements. The complimentary variables chosen for this experiment are position and momentum. The momentum is measured strongly after a weak measurement of the position. Hence the weak value obtained is going to be

$$\begin{aligned} \langle \pi_x \rangle_w &= \frac{\langle p|x\rangle\langle x|\Psi\rangle}{\langle p|\Psi\rangle} \\ &= \frac{e^{ipx/\hbar}\Psi(x)}{\Phi(p)} \end{aligned} \tag{35}$$

where $\Phi(p)$ is the momentum representation of the wavefunction and in case of post-selecting for $p = 0$, it would be a constant and equation (35) could be rewritten as

$$\langle \pi_x \rangle_w = k\Psi(x) \tag{36}$$

By performing the weak measurement for different values of x, one can directly obtain the information of the wavefunction on the measuring apparatus. At each point the position and the momentum of the pointer of the apparatus is shifted by an amount proportional to the real and imaginary part of the wavefunction. Previously in [11], an operational meaning of measurement of the real and imaginary parts of wavefunction was discussed. It was shown that letting $A_w = a + ib$, a weak measurement of pre- post-selected quantum system causes shifts both in position and momentum of the pointer given by

$$\langle q\rangle_f = \langle q\rangle_i + ga + gb\Big(m\frac{d}{dt}Var_q\Big). \tag{37}$$

$$\langle p\rangle_f = \langle p\rangle_i + 2gb(Var_p). \tag{38}$$

where Var_i is the variance of the variable i.

This experiment in particular was done using the transverse spatial wavefunction of a photo. As it is shown in figure (3), the experiment is performed in four stages: "preparation of the transverse wavefunction, weak measurement of the transverse position of the photon, post-selection of those photons with zero transverse momenta, and readout of the weak measurement."

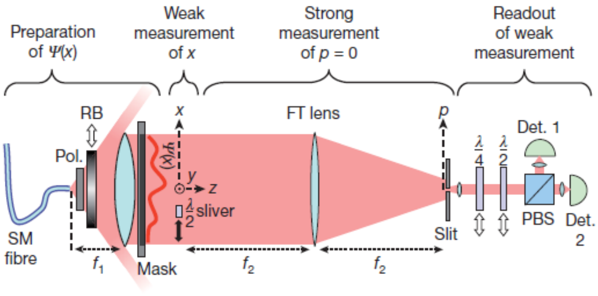

Figure 3: The diagram, illustrating the four stages of the experiment. [13]

In this experiment the transverse photon positions is weakly measured by coupling it to the photon's polarisation. Then one can use the method applied in section (2.2) to weakly measure the position of the photons. Then a Fourier transform lens followed by a single slit is used to post- select the photons with momentum $p = 0$. For the details of the experiment the reader might want to consult the original paper. [13]

Notwithstanding the validity of the equation (35), one may criticise the result of this experiment in the following sense. As Ref. ([17]) argues in its chapter (1.5.4), titled 'Wavefunction for Photons', it is not strictly speaking correct to treat photons exactly the same way as (non-relativistic) massive particles by writing the equation $\langle x|\Psi\rangle = \Psi(x)$ for them. Definition of $|x\rangle$ for photons is tricky and should not be thought of the same way as for massive particles. One might criticise this experiment as being the measurement of an electromagnetic field rather than reality of a wavefunction. Nevertheless, since equation (35) allows us to do the same for any wavefunction, this experiment should be reproducible for, say, electron, in which case give a new physical definition to wavefunction.

3 Protective measurement

In the last section we saw an example of measurement on an ensemble of pre- and post- selected quantum objects. This section investigates the possibility a type of measurement on a single particle and see if we can consider this as another method for measuring wavefunction and considering the wavefunction as a physical reality [1]. In describing protective measurements we again consider von Neumann interaction (1) to describe the measuring procedure. Nevertheless, we want to protect the wavefunction of the quantum system, $|\Psi\rangle$, such that the initial state $|\Psi(0)\rangle$, and final state $|\Psi(t)\rangle$ of the system are approximately the same and the entanglement in (6) never happens. If we succeed in protecting our quantum system in this ways, then the states will always evolve as a product state in the time interval of measurement interaction

$$|\alpha(0)\rangle|\Psi(0)\rangle \to |\alpha(t)\rangle|\Psi(t)\rangle \qquad t>0 \tag{39}$$

Using the interaction Hamiltonian and Schroedinger equation one sees

$$\frac{d}{dt}\langle\alpha(t)\Psi(t)|\pi|\alpha(t)\Psi(t)\rangle = -g(t)\langle\Psi(t)|A|\Psi(t)\rangle \tag{40}$$

hence one can determine the expectation value of the observable A by reading the change in the momentum of the pointer. Following the study of technical details of this method of protection we shall see whether if this type of measurement can be used to determine the wavefunction of a single system. There are two ways in which one can protect a quantum state, one by adding an extra protecting potential and taking the measurement in the adiabatic limits and the other through the Zeno effect. Both ways are described in the following. For the first let the initial state of the system be a non-degenerate eigenstate of the Hamiltonian. If the coupling function, $g(t)$, varies slowly enough as $T \gg \hbar/\Delta E$, where ΔE is the energy gap between $|\Psi\rangle$ and the next eigenstate, then by adiabatic theorem the system is required to stay at this state for the period of the interaction. Furthermore, the coupling is also weakened sufficiently to ensure that the initial and final state don't differ radically.

To protect the state using the Zeno effect, the state does not need to be an eigenstate of the Hamiltonian. The adiabatic limit is also unnecessary. Taking $|\Psi_0(t)\rangle$ as the evolution of the system under the free Hamiltonian of (1), H_0. Repeated measurement the observable $\hat{O}$, densely in time and such that $|\Psi_0(t)\rangle$ is a non-degenerate eigenstate of $\hat{O}$ causes only a negligible difference between the state of the system $|\Psi(t)\rangle$ and $|\Psi_0(t)\rangle$. This difference goes to zero as the number of the measurements goes to infinity.

To see how the first method (*i.e.* protection by additional potential) works, a modified Stern-Gerlach experiment was introduced in [1]. Here we revisit the argument followed by an analysis of the meaning of it.

Consider a Stern-Gerlach apparatus for spin-$\frac{1}{2}$ quantum systems in which particles pass through an inhomogeneous magnetic field $g(t)B_1(x)$ but is protected by a large external homogeneous magnetic field B_0 in z direction. The Hamiltonian which describes this interaction is now

$$H = -\mu B_0\sigma_z - \mu g(t)\sigma.B_1 \tag{41}$$

Given the initial state of the particle

$$\Psi(x,t) = \begin{pmatrix}1\\0\end{pmatrix}\exp\Big(i\frac{\mu B_0}{\hbar}t\Big), \qquad t>0 \tag{42}$$

Solving the Schrdinger equation gives

$$\begin{aligned}\Psi(x,t) = {}&\cos(\frac{\theta}{2})\exp\Big(i\frac{\mu B(x)}{\hbar}t\Big)\Psi_+(x,t)\\ &+\sin(\frac{\theta}{2})\exp\Big(-i\frac{\mu B(x)}{\hbar}t\Big)\Psi_-(x,t)\end{aligned} \tag{43}$$

where $\Psi_+(x,t)$ and $\Psi_-(x,t)$ are given by

$$\begin{aligned}\Psi_+ &= \begin{pmatrix}\cos\frac{\theta}{2}\\ \exp(i\phi)\sin\frac{\theta}{2}\end{pmatrix}\\ \Psi_- &= \begin{pmatrix}\sin\frac{\theta}{2}\\ -\exp(i\phi)\cos\frac{\theta}{2}\end{pmatrix}\end{aligned} \tag{44}$$

$B(x)$ is the total magnetic field, and θ is the polar angle between B and the z axis. ϕ represents the other polar angle.

In the absence of protecting magnetic field, the only contribution to the magnetic field is B_1 and hence by the end of the interaction wavefunction splits into non-overlapping parts. However, in the presence of large B_0 the angle θ becomes very small and (43) becomes

$$\Psi(x,t) = \exp\Big(i\frac{\mu}{\hbar}(B_0t + TgB_1(x)\cos\gamma)\Big)|0\rangle, \quad t>T \tag{45}$$

where γ is the angle between B_1 and B_0. In this case, there is no split of the wavefunction and no entanglement of type (6). However, equation (45) describes the momentum gained by the particle, to be approximately $\mu Tg\langle\nabla\{B_1(\mathbf{x})\cos\gamma)\}\rangle$. As demonstrated, there is no split of the wavefunction and no collapse. Therefore, in the limits of weak and adiabatic interaction one can repeat the measurement in different directions of the inhomogeneous magnetic field and hence determine the direction of B_0 and the spin of the system. This method enables us to directly measure the wavefunction of a quantum system as a physical object, extended over the space. Next section addresses this issue and criticisms in detail.

3.1 Protective measurements and the reality of wavefunction

In the last section we showed how the expectation value of an operator could be calculated for a single system under the protective condition. In this section we explicate how the wavefunction can be directly measured by measuring the density and quantum mechanical current at every point [6]. Let A_n be the projection operator on a region V_n with volume v_n. Within this region the operator is just $A_n = \frac{1}{v_n}$. Measurement of this operator gives

$$A_n = \frac{1}{v_n} \int_{V_n} \mid \Psi \mid^2 dv = \mid \Psi_n \mid^2, \tag{46}$$

where $|\Psi_n|^2$ is the average density in the region. By making the energy gap of the Hamiltonian sufficiently large and given the required time for the adiabatic process, one can determine the density of the wavefield without changing the wave, using protective measurement. In the general case, to determine the wavefunction, one requires to be able to measure the quantum mechanical current as well. Using the same projection operators A_n, one can measure the operator $B_n = \frac{1}{2i}(A_n \nabla + \nabla A_n)$. measuring the expectation value of B_n indeed gives the average quantum mechanical current

$$j = \frac{1}{2im}(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) \tag{47}$$

in the same region defined by V_n. Decomposing the wavefunction into phase and amplitude,

$$\Psi(x) = R(x) \exp(iS(x)), \tag{48}$$

one can determine the phase of the wavefunction at any point by directly measuring current and density,

$$j(x)/\rho(x) = \frac{\nabla S}{m} \tag{49}$$

This argument demonstrates that the stationary state can be observed, and hence is physical. Nonetheless, one can see some limitations in assigning physicality to a wavefunction in this method. Some of these limitation was pointed out in the original paper by Aharonov and Vaidman [1], and the others were criticised in other publications [3, 15]. In the following, we shall briefly review the major criticisms.

Adiabacity of the process requires that the measurement to be done in a long period of time. One may be inclined to believe that this is now a time-averaging of the physical properties of the system as opposed to the properties of the wavefunction at a given time during the measurement. Nevertheless, as shown in last section, the wavefunction of the system under protective measurement does not change significantly during the process. Hence the time-averaging of physical properties cannot cause small changes in the pointer of the measuring device, which add up and give the final result of the protective measurement.

Perhaps the most common argument for criticising protective measurements is that the process cannot be applied to an arbitrary unknown quantum state. Because one must know the state of the system to be able to protect it (e.g. in the modified Stern-Gerlach experiment we could not know which direction to apply the homogeneous magnetic field to protect the spin wavefunction). However, this argument is misguided. To perform protective measurement, the physicist only requires to know that the quantum state is protected. Then she can measure the density and quantum current of the wavefunction, without knowing the wavefunction and how it is protected. The main point is that in many circumstances protection is done by nature or the physics of problem. A common example happens in quantum computing. In Quantum Annealing one knows that the state of the system is the ground state of the Hamiltonian and because of the adiabatic theorem it will remain a ground state during the time evolution of the Hamiltonian. Nevertheless, that ground state is practically impossible to calculate, due to the lack of sufficient computational resources.

And finally the last major argument against the validity of protective measurement is that the protection method may make one to think that protective measurement measures the properties of the protection potential, as opposed to the physical properties of the quantum system. This belief could be, mistakenly, become stronger if one looks at some of the proposed thought experiments, such as the modified Stern-Gerlach experiment, in which there was a direct correspondence between the direction of the strong homogeneous magnetic field (the protection) and the direction of the spin of the system. However, this idea cannot be true. There are many other examples in which there are no such correspondence. And more importantly, one can confirm that in protective measurement of a wavefunction, different protection methods lead to the same results. For instance in the protection Hamiltonian

$$H = G_0(1 - |\Psi_0\rangle\langle\Psi_0|) + \sum_i G_i |\Psi_i\rangle\langle\Psi_i| \tag{50}$$

where the first term penalises the possible degenerate states of the Hamiltonian and the second term ensures that the gap of the Hamiltonian is large enough for the adiabatic measurement, the orthogonality of the ground state $|\Psi_0\rangle$ and the excited states $|\Psi_i\rangle$ is sufficient for the protection, and the protected ground state could be protectively measured. Nonetheless, one might choose to say although we are not measuring any potential, but we are measuring a certain property of the Hamiltonian, being "the property to have an eigenstate this quantum state characterised by the expectation value of various operators." Even adopting this view, we have measured the same amount of information one could have from knowing the Schroedinger wave of Ψ and is an equivalent task.

So far we have applied protective measurement method to study the nature of wavefunction. In doing that, we have seen how one can measure density and quantum mechanical current, without changing the wavefunction. Being equipped with these tools, one may hope to be able to test quantum theories of motion (*e.g.* Bohmian mechanics), discussed in the next section.

3.2 Testing Bohmian trajectories by protective measurements and criticisms

Bohmian model is a theory of quantum mechanics, experimentally equivalent to the standard quantum theory, but with the only additional assumption that particles exist, even when no one looks at them [7]. Therefore, a complete description of the state of a system is given by the wavefunction, $\Psi(x,t)$, and the position of the system, $X(t)$, at any time. Briefly Bohmian mechanics follows from four postulates:

- The complete description of an isolated system is give by its wave function, $\Psi(x,t)$, and the particle's configuration, $X(t)$.
- Evolution of the wavefunction obeys the time dependent Schroedinger equation.
- Evolution of particle's position is given by the guidance equation $\frac{d}{dt}x = \frac{\nabla S}{m}$, where S is the phase function.
- At time t, the particle will be in a region $[x\ ,\ x+\delta x]$ with probability $R^2(x,t)d^3x$ where $R^2 =\mid \Psi \mid^2$.

One can see that the guidance equation is just the quantum current over the density as in equation (49). Hence this is not a new equation of motion for quantum mechanics. The only additional assumption is that particles have positions all the time. Therefore, while the third postulate clarifies the equation of motion of the particle, it would suffice to replace it with only one sentence "particles move continuously". It worth mentioning that the measurement problem discussed in section (1) does not appear in this model, due to the denial of representational completeness. For the details see [10].

Substituting the phase-amplitude decomposed wavefunction (48) into the Schroedinger equation and separating the real and imaginary parts gives the two coupled equations,

$$\frac{\partial R^2}{\partial t} + \nabla.(\frac{R^2\nabla S}{m}) = 0 \tag{51}$$

and

$$-\frac{\partial S}{\partial t} = \frac{(\nabla S)^2}{2m} + V + Q \tag{52}$$

where Q is given by

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}. \tag{53}$$

Comparing the equation (51) with (49) one can see the (51) is the continuity equation. Equation (52) is the Hamilton-Jacobi equation for quantum particles. One may notice that this equation is different from the classical one due to the last term, Q, in (52). This is called quantum potential. Hence, according to this picture, a particle feels a force due to both the gradients of quantum and classical potentials. Therefore, while particles have trajectories, they are different from those of Newtonian mechanics.

Having the model introduced, now we can review the proposal by Aharonov, Engler and Scully [5] to test Bohm trajectories using protective measurements.

Let a particle of mass m, moving along the x axis, be measured under the protective conditions. This particle is confined to a box with sides of length $2l$ and before the measurement process is in its ground state with energy E. The measurer has mass M and is moving along the X axis and is coupled to the quantum particle under the measurement by the interaction

$$H_{int} = \epsilon\frac{\hbar}{T}f(t/T)\delta(x)X, \tag{54}$$

where ϵ defines the strength of the interaction, and T is the duration of the measurement. The delta function is supposed to ensures that the interaction is highly localised and the pointer will be shifted only by presence of the particle to a region very close to $x = 0$.

AES argue that a) the measuring device only weakly and adiabatically perturbs the particle's state, the measurement will not disturb the Bohmian trajectories, and b) one can calculate that the majority of Bohmian trajectories of the particle never come near the interaction region (details of the calculation can be found in [5]). This is a challenge when considering the trajectories as physically real because one can see the shift in the pointer state and this shift with the assumption of the reality of trajectories require them to be near the interaction region. They suggest that one can only think of the trajectories as mathematical constructions with no ontological meaning corresponding to the particle's position.

One might be ready to accept the AES argument based on intuition. Nevertheless, we shall see through calculations of the quantum potential (53) of the combined system that this is not a challenge for the reality of Bohmian trajectories. One can confirm form [8] that the quantum potential of the total system, $Q(x,X,t)$ under the adiabatic approximation is

$$Q(x,X,t) \approx -\frac{\hbar^2}{2m}\frac{\nabla^2\Psi_\gamma(x)}{\Psi_\gamma(x)} + G(X,t) \tag{55}$$

where $\Psi_\gamma(x)$ is the ground state of the quantum system evolving under the adiabatic limits and the explicit form of the $G(X,t)$ is irrelevant. Since the wavefunction of the system has no nodes and is factorisable, this is an acceptable form of quantum potential.

Furthermore, from the Schroedinger equation of the system we have

$$-\frac{\hbar^2}{2m}\nabla^2\Psi_\gamma(x) + f(t)\delta(x)X\Psi_\gamma(x) \approx E(X,t)\Psi_\gamma(x) \tag{56}$$

using (55) and (56) the quantum potential can be written as

$$Q(x, X, t) \approx -f(t)\delta(x)X + E(X, t) + G(X, t). \qquad (57)$$

Since the effective potential which causes the force on Bohmian particle is $U = H + Q$, the delta function gets cancelled. Hence, despite the localised interaction term in the Hamiltonian, the system acts non-locally and the particle does not have to come in the vicinity of the pointer to influence it. Hence there is nothing inconsistent in the Bohmian picture of this problem. AES believe that "[it] should have been quite disturbing to adherents of Bohmian mechanics because it implies that the Bohm trajectories are forever hidden." Nevertheless, the Bohmian mechanics is known to be a *hidden variable* theory, and the adherent of the this theory surely accepts the unknowability of the Bohmian trajectories. Nonetheless, it is possible that the theory of weak measurement may in future give us clue on how to measure properties of empty waves in the framework of Bohm. This should not be surprising from an experiment in which one prepares a quantum wavefunction which includes nodes and then weakly scans the wavefield, followed by a strong measurement of the position. We know that the Bohmian particle cannot cross the nodes, and we also know that the measurement is weak and cannot change the wavefield. Hence the weak values must refer to the properties of the field, not the particle. Of course all of these depend on admission of the premisses of Bohmian model.

4 Conclusion and future paths

Weak measurement is indeed an important new tool to seek the answer to problems in quantum foundations which previously seemed to be untouchable in the framework of physics. A lot of issues which used to be a matter of philosophy are now testable in laboratories. In particular weak measurement gives us clue that wavefunction represents a physically real entity. Nevertheless, the type of experiments on the reality of wavefunction are still very young and unsettled. For instance the experiment discussed in section (3.1) could be criticised for semi-classicality of photonic experiments, although the theory behind it is undoubtedly correct. Probably the best way of ensuring that the results are actually properties of the wavefunction is performing the experiments on other quantum system (such as electrons), which is an interesting experiment to do in the future.

There are also very deep conceptual issues within the framework of weak measurements. Interpretation of the negative and complex weak values are among them.

It is also an interesting problem to see if weak measurements can eventually make a physical distinction between the standard quantum mechanics and other models of quantum theory which are experimentally equivalent to the standard one, such as the Bohmian model. After all these theories are different until the stage of the basic measurement, even in the case of single particle. Such a distinction through weak measurements should not be more surprising than the results obtained in the weak measurement solution of Hardy's paradox about counter-factual reasoning.

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